In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x, as a function of t, where t is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + Ce^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^{+}.$$

The researchers decide to let $L = 200$.

(i) State the assumption being made in assuming $L = 200$. [1]

At $t = 0$, the population of wolves is 40.

(ii) Find the value of C. [2]

At $t = 5$, the population of wolves is found to have increased to 70.

(iii) Find the value of k. [2]

(iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{dx}{dt} = ax\ln\left(\frac{200}{x}\right), \ a \in \mathbb{R}^{+}.$$

(i) Write down the value of $\frac{dx}{dt}$ when $x = 200$. [1]

(ii) Interpret your answer to part (b)(i) in context. [1]

Consider the function $f(x) = \ln(\ln 200 - \ln x)$, where $0 < x < 200$.

(iii) Show that $f'(x) = \dfrac{-1}{x\ln\left(\dfrac{200}{x}\right)}$. [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{dx}{dt} = ax\ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - Ae^{-at},$$

where A is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at $t = 0$ to find the value of A. Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^{+}$. [2]

(vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$. Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{dx}{dt} = 0.0855x\ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$. Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]

a)

(i) stable long term population would be 200

(ii) $40 = \frac{200}{1+Ce^0} \Rightarrow 1+C = \frac{200}{40}$

$C = 4$

(iii) $70 = \frac{200}{1+4e^{-k(5)}}$

$1 + 4e^{-5k} = \frac{20}{7}$

$e^{-k5} = \left(\frac{20}{7} - 1\right)\frac{1}{4}$

$-5k = \ln \frac{13}{7\times 4} = \ln\left(\frac{13}{28}\right)$

$k = 0.1535$

10 $x = \frac{200}{1+4e^{-0.1535(10)}}$

$= 107.42$

b $\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right)$

(i) $\frac{dx}{dt} = a \cdot 200 \ln(1) = 0$

(ii) when population as reached 200 its stays stable with time.

(iii) $f(x) = \ln(\ln 200 - \ln x)$.

$f'(x) = \frac{1}{\ln 200 - \ln x} \cdot \left(-\frac{1}{x}\right)$

$= \frac{-1}{x \cdot \ln\left(\frac{200}{x}\right)}$

(iv) $\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right)$

$\frac{1}{x \ln\left(\frac{200}{x}\right)} dx = a\,dt$

Integrating both sides.

$\int \frac{dx}{x \ln\left(\frac{200}{x}\right)} = \int a\,dt$

from previous part :-

$-\ln(\ln 200 - \ln x) = at + C$

$\ln(\ln 200 - \ln x) = -at - C$

$\ln 200 - \ln x = e^{-at-C} = Ae^{-at}$

$A = e^{-C}$

$= \ln x = \ln 200 - Ae^{-at}$

v) $\ln x = \ln 200 - Ae^0$

$A = \ln 200 - \ln x$

$A = \ln 200 - \ln 40$

$= \ln 5$

vi) $\ln 70 = \ln 200 - \ln 5 . e^{-a(5)}$.

$\ln 5 . e^{-5a} = \ln\left(\frac{200}{70}\right)$

$e^{-5a} = 0.65221$

$-5a = \ln 0.65221$

$a = 0.085477$

$\Rightarrow a = 0.0855$

(vii) $\ln x = \ln 200 - \ln 5 . e^{-0.0855 \times 10}$

$\ln x = 4.61385$

$x = 100.8667$

≈ 101

C Second method is more closure to 85 (actual value after 10 years).

d/i Euler's method.

t_0	t_1	x
	5	70
5	5.5	72.09159
	6	74.159
	6.5	76.187
	7	:
	7.5	:
	8	:
	8.5	:
	9	:
	9.5	87.651644
	10	89.4280354

use $h = 0.5$

$x_0 = 70$

$x = x_0 + 0.5\left(\frac{dy}{dt}\right)$

It gives more accurate results than other methods.