Shreeji Education

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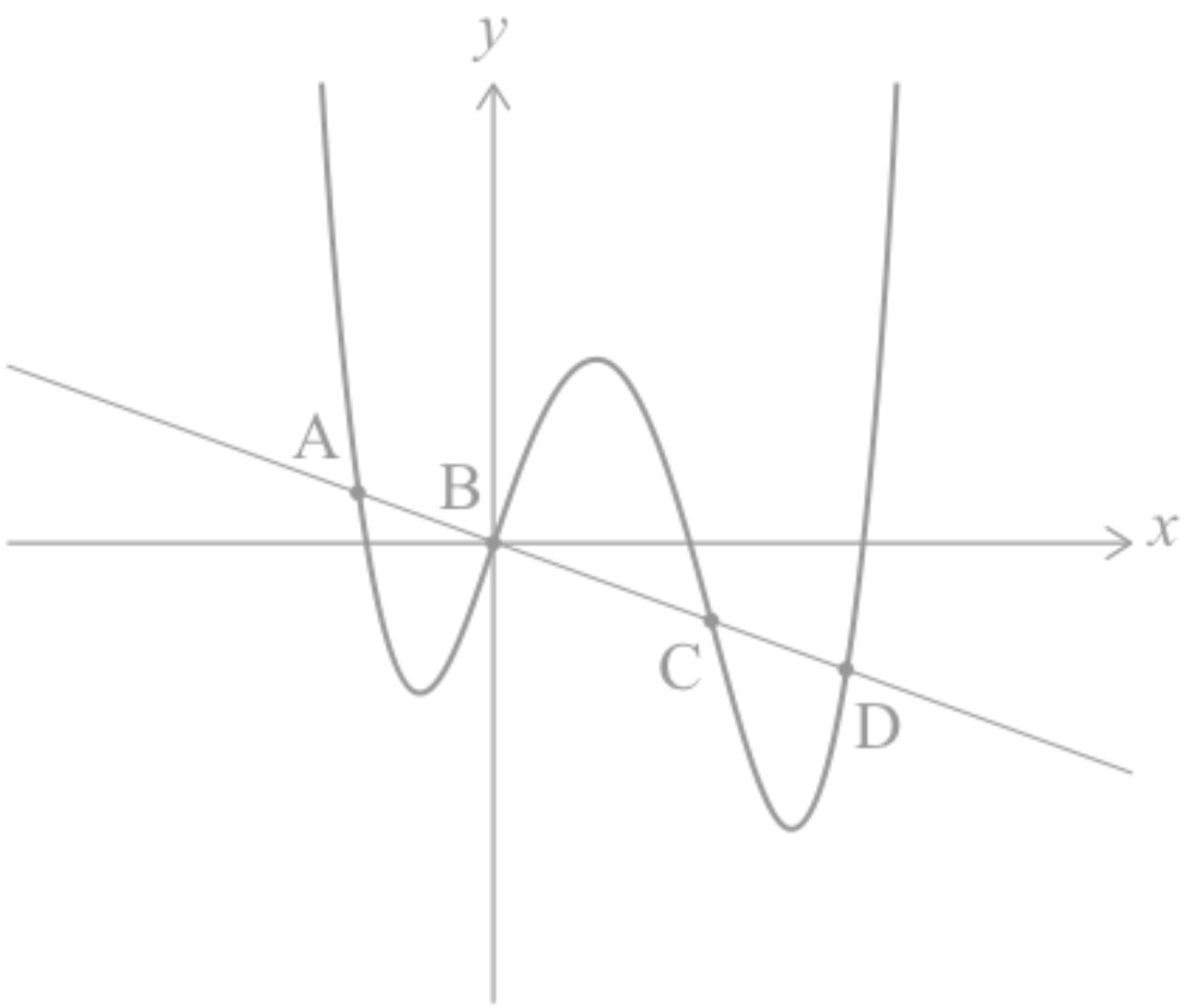
AA HL -P3

Question 1

Reference - IBDP - past papers

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



(a) Find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$. [3]

Click to go to solution slide

(b) Find the coordinates of B and C. [4]

(c) Show that the equation of the line through B and C is $y = -0.375x$. [2]

(d) Find, correct to three decimal places, the x-coordinate of D. [2]

Click to go to solution slide

Now consider the general curve $y = x^4 - mx^3 + nx$, where m, $n \in \mathbb{R}$ and $m > 0$.

(e) Find the x-coordinates of the two points of inflexion in terms of m. [3]

Click to go to solution slide

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x-coordinate of point A, and similarly for x_B, x_C and x_D. It is given that $x_A < x_B < x_C < x_D$.

(f) (i) Write down the coordinates of B. [1]

(ii) Find, in terms of m and n, the coordinates of C. [2]

(g) Show that the equation of the line through B and C is $y = \left(-\frac{m^3}{8} + n\right)x$. [2]

Click to go to solution slide

(h) Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. [7]

(i) Hence, find the exact value of $\frac{x_B - x_A}{x_C - x_B}$. [2]

Click to go to solution slide

a) $\frac{dy}{dx} = 4x^3 - 9x^2 + 3$

$\frac{d^2y}{dx^2} = 12x^2 - 18x$

Go to next question

Go to same question

b) $12x^2 - 18x = 0$

$\Rightarrow \quad x = 0, \; x = 3/2$

$x = 0$

$y = 0$

$x = 3/2$

$y = -0.5625$

$B \rightarrow (0, 0)$

$C \rightarrow (3/2, -0.5625)$

c) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 + 0.5625}{0 - 3/2}$

$= -0.375$

eq: $y - y_1 = m(x - x_1)$

$y - 0 = -0.375(x - 0)$

$\Rightarrow \quad y = -0.375x.$

$\dfrac{dy}{dx} = 4x^3 - 9x^2 + 3$

$-0.375 = 4x^3 - 9x^2 + 3.$

using GDC $x = 2.427$.

In gdc (if solve does not give, plot with $y = 4x^3 - 9x^2 + 3.375$ then the x intercept will give.

Go to same question

Go to next question

c

$$y = x^4 - mx^3 + nx$$

$$y' = 4x^3 - 3mx^2 + n$$

$$y'' = 12x^2 - 6mx$$

$$y'' = 0 \Rightarrow 6x(2x - m) = 0$$

$$\Rightarrow x = 0, \quad x = m/2$$

Go to same question

Go to next question

f/i $\quad x = 0 \quad y = 0 \Rightarrow B(0,0)$

f/ii $\quad y = \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}$

$y = \frac{-m^4}{16} + \frac{nm}{2}$

$\Rightarrow$

$C \to \left(\frac{m}{2}, \frac{-m^4}{16} + \frac{nm}{2}\right)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{-m^4}{16} + \frac{nm}{2} - 0}{\frac{m}{2} - 0}$

$= \frac{-m^4 \cdot 2}{16 \times m} + \frac{nm \cdot 2}{2m}$

$= \frac{-m^3}{16} + n$

$\Rightarrow y_2 - y_1 = m(x_2 - x_1)$

$\Rightarrow y - 0 = \frac{-m^3}{16} + n \, (x - 0)$

$\Rightarrow y = \left(\frac{-m^3}{16} + n\right) x$

Go to same question

Go to next question

using $y = x^4 - mx^3 + nx$

$$\Rightarrow \left(-\frac{m^3}{8} + n\right)x_A = x_A^4 - m x_A^3 + n x_A$$

$$\Rightarrow x_A^4 - m x_A^3 + \cancel{n x_A} + \frac{m^3}{8} x_A - \cancel{n x_A} = 0$$

$$\Rightarrow x_A^4 - m x_A^3 + \frac{m^3}{8} x_A = 0$$

$$\Rightarrow x_A\left(x_A^3 - m x_A^2 + \frac{m^3}{8}\right) = 0$$

Sum of roots is $x_A + x_B + x_C = m$

$(-b/a)$

Product of roots is $x_A \cdot x_B \cdot x_C = -\frac{m^3}{8}$

$\left\{ (-1)^n \frac{c}{a} \right\}$

$$X_C = m/2 \Rightarrow X_A + X_B = m - m/2 = m/2$$

$$\Rightarrow X_B = \frac{m}{2} - X_A$$

$$\Rightarrow X_A \cdot \left(\frac{m}{2} - X_A\right) \cdot \frac{m}{2} = -\frac{m^3}{8}$$

$$\Rightarrow X_A \left(\frac{m}{2} - X_A\right) = -\frac{m^2}{4}$$

$$X_A \frac{m}{2} - X_A^2 = -\frac{m^2}{4}$$

$$\text{or } X_A^2 - X_A \cdot \frac{m}{2} - \frac{m^2}{4} = 0$$

Go to same question

Go to last part of solution

$$\Delta = b^2 - 4ac$$
$$= \frac{m^2}{4} - 4\cdot 1\cdot\left(-\frac{m^2}{4}\right)$$
$$= \frac{m^2}{4} + m^2 = \frac{5m^2}{4}.$$

$$x_A = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{\frac{m}{2} \pm \sqrt{\frac{5m^2}{4}}}{2}$$

$$x_A < x_B < x_C$$

$$\Rightarrow x_A = \frac{\frac{m}{2} - \sqrt{5}\cdot\frac{m}{2}}{2}$$

$$x_A = \frac{m}{4} - \sqrt{5}\cdot\frac{m}{4}$$

$$\frac{x_B - x_A}{x_C - x_B} = \frac{0 - \frac{m}{4} + \frac{\sqrt{5}\cdot m}{4}}{\frac{m}{2} - 0}$$

$$= -\frac{m\cdot 2}{4\cdot m} + \frac{\sqrt{5}\,m\cdot 2}{4\cdot m}$$

$$= -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ or } \frac{\sqrt{5}-1}{2}$$

Go to question

End