SECTION A

Q1

$T_7 = 6 \Rightarrow a + 6d = 6 \quad \text{--- (1)}$

$T_6 + T_{12} = 24$

$\Rightarrow a + 5d + a + 11d = 24$

$\Rightarrow 2a + 16d = 24$

$\Rightarrow a + 8d = 12 \quad \text{--- (2)}$

Solving $6 - 6d = 12 - 8d$

$\Rightarrow 2d = 6 \quad d = 3$

and $a = -12$ } Ans.

Q2 $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

a

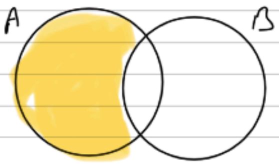
$P(A \cap B) = \dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{12}$

b $P(A|B') = \dfrac{P(A \cap B')}{P(B')}$

$P(B') = \dfrac{2}{3}$ $\quad P(A \cap B') = P(A) - P(A \cap B)$

$= \dfrac{1}{2} - \dfrac{1}{12} = \dfrac{5}{12}$

$\Rightarrow P(A|B') = \dfrac{5 \cdot \cancel{3}}{\cancel{12}_4 \cdot 2} = \dfrac{5}{8}$ Ans



Q3

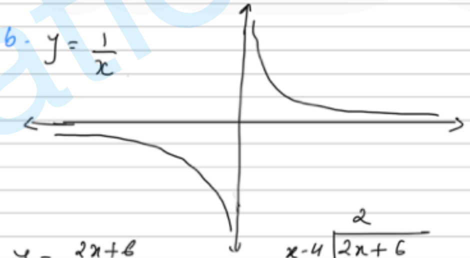
a. $1 = \dfrac{-10A + B}{-14} \Rightarrow -14 = -10A + B$

$-12 = \dfrac{3A + B}{-1} \Rightarrow 12 = 3A + B$

$\Rightarrow -14 + 10A = 12 - 3A$

$\Rightarrow 13A = 26$ $\quad A = 2$ and $B = 6$ } Ans

b. $y = \dfrac{1}{x}$

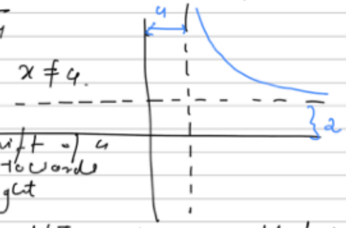


$y = \dfrac{2x+6}{x-4}$

$$\begin{array}{r} 2 \\ x-4\overline{)\,2x+6} \\ 2x-8 \\ \hline 14 \end{array}$$

$y = 2 + \dfrac{14}{x-4}$

$x - 4 \neq \Rightarrow x \neq 4$



$\therefore$ vertical shift $\rightarrow$ 4 unit towards right

Horizontal asymptote $= y = 2$, graph has shifted upwards.

$\therefore$ translation: $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

and vertical stretch is 14.

So vertical stretch 14 } $\rightarrow$ followed by translation $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Q4

$3\log_8 10x - \log_4 x = 1$ for $x > 0$.

$3\log_{2^3} 10x - \log_{2^2} x = \log_2 2$

$\Rightarrow 3\left(\frac{\log_2 10x}{\log_2 2^3}\right) - \frac{\log_2 x}{\log_2 2^2} = \log_2 2$

$\Rightarrow \cancel{3}\frac{(\log_2 10x)}{\cancel{3}\log_2 2} - \frac{\log_2 x}{2\log_2 2} = \log_2 2$

$\Rightarrow \log_2 10x - \frac{1}{2}\log_2 x = \log_2 2$

$\Rightarrow \log_2 10x - \log_2 \sqrt{x} = \log_2 2$

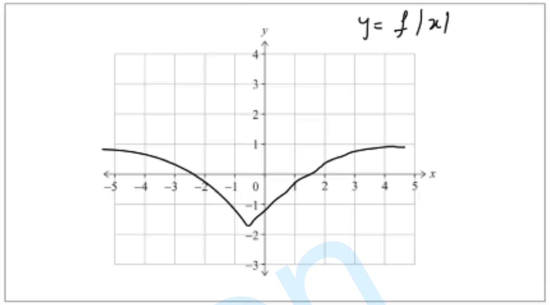
$\Rightarrow \log_2\left(\frac{10x}{\sqrt{x}}\right) = \log_2 2$

$\Rightarrow \frac{10x}{\sqrt{x}} = 2 \Rightarrow 10\sqrt{x} = 2$

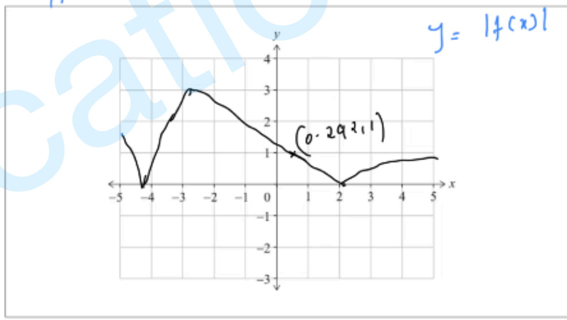
$\Rightarrow \sqrt{x} = \frac{1}{5}$

[as $x > 0$] $\Rightarrow x = \frac{1}{25}$ Ans

Q5 a/i



a/ii



b $\quad |f(x)| \geq (f(x))^2$

$-f(x) \geq (f(x))^2 \geq f(x)$

from graph: for $0.292 \leq x \leq 5$
$-f(x) \geq (f(x))^2$ Ans

and for $-4.93 \leq x \leq -4.27$
$(f(x))^2 > f(x)$. Ans

Note: in this region $-1 \leq f(x) \leq 1$
$\therefore (f(x))^2 \leq 1$.

eg: $0.25 > (0.25)^2$

Q6

$$\int x\,\text{cosec}^2 x\,dx = \ln|\sin x| - x\cot x + C$$

$$x\frac{dy}{dx} + 3y = \frac{\text{cosec}^2 x}{x}$$

of the form: $\frac{dy}{dx} + P(x)y = Q(x)$

with $IF = e^{\int P(x)\,dx}$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\text{cosec}^2 x}{x^2}, \quad P(x) = \frac{3}{x}.$$

$$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

$$\therefore \int\left(x^3\frac{dy}{dx} + 3x^2y\right)dx = \int x\,\text{cosec}^2 x\,dx$$

$$\Rightarrow x^3 y = \ln|\sin x| - x\cot x + C$$

$$\Rightarrow y = \frac{1}{x^3}\left\{\ln|\sin x| - x\cot x + C\right\}$$

$$\frac{8}{\pi^2} = \frac{8}{\pi^3}\left\{\ln\left|\sin\frac{\pi}{2}\right| - \frac{\pi}{2}\cot\frac{\pi}{2} + C\right\}$$

$$\Rightarrow \pi = \ln(1) - \frac{\pi}{2}(0) + C$$

$$\Rightarrow \pi = 0 - 0 + C \quad \Rightarrow C = \pi$$

$$\therefore y = \frac{1}{x^3}\left\{\ln|\sin x| - x\cot x + \pi\right\}$$

Q7

a

$0.6 - 2a \geq 0 \quad 0.3 \geq a \quad a \leq 0.3$

$3a \geq 0 \quad \Rightarrow \quad a \geq 0 \quad a \geq 0$

$0.4 - a \geq 0 \quad \Rightarrow \quad 0.4 \geq a \quad a \leq 0.4$

$\therefore 0 \leq a \leq 0.3$

b $a = 0.2$

x	1	2	3
$P(x)$	0.2	0.6	0.2

$E(X) = 0.2 + 1.2 + 0.6 = 2$

$Var(2-X) = Var(X)$

$\sum x^2 P(x) = 0.2 + 2.4 + 1.8$
$= 4.4$

$Var(X) = Var(2-X) = \sum x^2 P(x) - [E(X)]^2$
$= 4.4 - 4$
$= 0.4$

Q8 $T = T_1 + T_2$

a) $T = \frac{d_1}{s_1} + \frac{d_2}{s_2}$

$= \frac{400 \sec\theta}{0.8} + \frac{1000 - 400\tan\theta}{1.2}$

$= 500\sec\theta + \frac{5}{2}\left(\frac{1000 - 400\tan\theta}{3}\right)$

$\left\{ 1.2 = \frac{12}{10} = \frac{6}{5} = \frac{3\times 2}{5} \right.$

$= 500\sec\theta + \frac{2500 - 1000\tan\theta}{3}$

b $\frac{dT}{d\theta} = 500\sec\theta\tan\theta + 0 - \frac{1000}{3}\sec^2\theta$

c using optimization for minimum value

$\frac{dT}{d\theta} = 0 \Rightarrow$

$500\sec\theta\tan\theta - \frac{1000}{3}\sec^2\theta = 0$

$\Rightarrow 500\sec\theta\left(\tan\theta - \frac{2\sec\theta}{3}\right) = 0$

$\Rightarrow 500\sec\theta = 0$ or $\tan\theta = \frac{2\sec\theta}{3}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{2}{\cos\theta \cdot 3}$

$\Rightarrow \sin\theta = \frac{2}{3}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{2}{3}\right)$

Now $\sin\theta = \frac{opp}{hyp} = \frac{PX}{400\sec\theta}$

$\Rightarrow \frac{2}{3} = \frac{PX}{400\sec\theta}$

$PX = \frac{800\sec\theta}{3}$

$\left\{ \begin{array}{l} \sin\theta = 2/3 \\ \cos\theta = \sqrt{1-\frac{4}{9}} \\ = \sqrt{5}/3. \end{array} \right.$

$PX = \frac{800}{3}\frac{3}{\sqrt{5}}$

$\sec\theta = \frac{3}{\sqrt{5}}.$

$= \frac{800 \times \sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \frac{800\sqrt{5}}{5}$

$= 160\sqrt{5}$ Ans.

SECTION B

Q9 a. $f(x) = 0 \Rightarrow \frac{x^2}{2} + kx + 13 = 0.$

no real roots $\Rightarrow \Delta < 0.$

$\Rightarrow b^2 - 4ac < 0$

$k^2 - 4 \cdot \frac{1}{2} \cdot 13 < 0$

$\Rightarrow k^2 - 26 < 0$

$k^2 < 26 \qquad k \in \mathbb{Z}^+$

$\Rightarrow \underline{k \leq 5}$

b/i $f(x) = \frac{x^2}{2} + 5x + 13.$

axis of symmetry is $x = \frac{-b}{2a}$

$x = \frac{-5}{2 \cdot \frac{1}{2}} = -5$

so $\underline{\underline{x = -5}}$

b/ii $y = \frac{25}{2} - 25 + 13$

$= \frac{25 - 50 + 26}{2}$

$y = \frac{1}{2}$ or 0.5

co-ordinate $(-5, 0.5).$

c $\frac{dy}{dx} = \frac{1}{2}(2)x + 5 = x + 5$

$\left(\frac{dy}{dx}\right)_{x=-3} = -3 + 5 = 2$

$\therefore$ slope of normal $= -1/2 \qquad \{m_1 m_2 = -1\}$

eq: of normal

$y - y_1 = m(x - x_1)$

$y - \frac{5}{2} = \frac{-1}{2}(x + 3).$

$y - \frac{5}{2} = -\frac{x}{2} - \frac{3}{2}$

$y = -\frac{x}{2} - \frac{3}{2} + \frac{5}{2}$

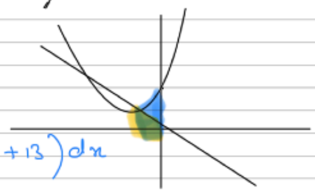
$y = -\frac{x}{2} + 1$

$\left\{ y = \frac{9 - 15}{2} + 13 \right.$

$= \frac{9}{2} - 2$

$= \frac{5}{2}$

d. shaded region is from



$\int_{-3}^{0} \left(\frac{x^2}{2} + 5x + 13\right) dx - \int_{-3}^{0} \left(-\frac{x}{2} + 1\right) dx$

$= \left[\frac{x^3}{6} + \frac{5x^2}{2} + 13x\right]_{-3}^{0} - \left[-\frac{x^2}{4} + x\right]_{-3}^{0}$

$= -\left[-\frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}} + \frac{45}{2} - 39\right] - \left(0 - \left(-\frac{9}{4} - 3\right)\right)$

$= -\left(\frac{-9 + 45 - 78}{2}\right) - \frac{21}{4}$

$= -\left(\frac{-42}{2}\right)$

$= \frac{84 - 21}{4} = \frac{63}{4}$ Ans

Q10

a) $r = \begin{pmatrix}-1\\1\\-13\end{pmatrix} + \lambda\begin{pmatrix}7\\1\\2\end{pmatrix}$

b) direction $\vec{AB} = \begin{pmatrix}7-2\\-6+4\\1-2\end{pmatrix} = \begin{pmatrix}5\\-2\\-1\end{pmatrix}$

$s = \begin{pmatrix}2\\-4\\2\end{pmatrix} + \mu\begin{pmatrix}5\\-2\\-1\end{pmatrix}$

c) direction vectors are not integral multiples hence l_1 and l_2 are not parallel.

$\underbrace{\begin{array}{l} x = -1 + 7\lambda \\ y = 1 + \lambda \\ z = -13 + 2\lambda \end{array}}_{L_1}$ $\quad$ $\underbrace{\begin{array}{l} x = 2 + 5\mu \\ y = -4 - 2\mu \\ z = 2 - \mu \end{array}}_{L_2}$

Solving: $-1 + 7\lambda = 2 + 5\mu \Rightarrow 7\lambda - 5\mu = 3$

$1 + \lambda = -4 - 2\mu \Rightarrow \lambda + 2\mu = -5$

$\Rightarrow 7\lambda - 5\mu = 3$

$\quad 7\lambda + 14\mu = -35$

$\quad -19\mu = 38$

$\quad \mu = -2$

$\Rightarrow \lambda = -5 - 2\mu = -5 - 2(-2) = -5 + 4 = -1$

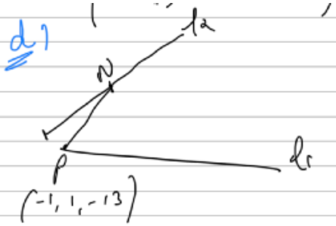
Plugging in third equation to check for intersection.

LHS: $-13 + 2(-1) = -15$

RHS: $2 - \mu = 2 + 2 = 4$ $\Big\}$ not equal.

$\therefore$ Lines are not parallel nor intersecting, hence they are skew.

d)



$\vec{PN} = \begin{pmatrix}2+5\mu+1\\-4-2\mu-1\\2-\mu+13\end{pmatrix} = \begin{pmatrix}3+5\mu\\-5-2\mu\\15-\mu\end{pmatrix}$

$\vec{PN}\cdot\vec{AB} = \begin{pmatrix}3+5\mu\\-5-2\mu\\15-\mu\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\-1\end{pmatrix}$

$= 15 + 25\mu + 10 + 4\mu - 15 + \mu$

$= 10 + 30\mu$

e) for closest point $\vec{PN} \perp l_2$

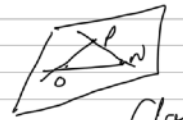
$\Rightarrow \vec{PN}\cdot\vec{AB} = 0$

$\Rightarrow 10 + 30\mu = 0 \Rightarrow \mu = -1/3$.

$N \begin{pmatrix}2\\-4\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}5\\-2\\-1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2-5/3\\-4+2/3\\2+1/3\end{pmatrix}$

$= \begin{pmatrix}1/3\\-10/3\\7/3\end{pmatrix}$

f)



direction of plane is a vector $\perp$ to $\vec{ON}$ and $\vec{OP}$ (for easy purpose. It can also be $\vec{PN}$ with any one).

$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -13 \\ 1 & -10 & 7 \end{vmatrix}$

Here we are considering only direction purpose so $\frac{1}{3}\begin{pmatrix}1\\-10\\7\end{pmatrix}$ has main direction $\rightarrow \begin{pmatrix}1\\-10\\7\end{pmatrix}$

$\vec{d} = +\hat{i}(7 - 130) - \hat{j}(-7 + 13) + \hat{k}(10 - 1)$

$= -123\hat{i} - 6\hat{j} + 9\hat{k}$.

equation of plane

ie $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$.

$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-123\\-6\\9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-123\\-6\\9\end{pmatrix}$

$\Rightarrow -123x - 6y + 9z = 0$.

Q11 a) $\omega = 1 + i\sqrt{3}$.

i. $r = \sqrt{1+3} = 2$.

$\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$.

$\Rightarrow \omega = 2e^{\pi/3\, i}$ ——— Ans.

ii $(1+i\sqrt{3})^8 + (1-i\sqrt{3})^8$

$= \left(2e^{\pi/3\, i}\right)^8 + \left(2e^{i - \pi/3}\right)^8$

$= 2^8 e^{8\pi/3\, i} + 2^8 e^{-8\pi/3\, i}$

$= 2^8 \left(e^{8\pi/3\, i} + e^{-8\pi/3\, i}\right)$.

$= 2^8\left(\cos\frac{8\pi}{3} + i\sin 8\pi/3 + \cos\left(-\frac{8\pi}{3}\right) + i\sin\left(-\frac{8\pi}{3}\right)\right)$

$= 2^8\left(\cos 8\pi/3 + \cancel{i\sin 8\pi/3} + \cos 8\pi/3 - \cancel{\sin 8\pi/3\, i}\right)$

$= 2^8(2\cos 8\pi/3)$

$= 2^8\left(2\cos\left(3\pi - \frac{\pi}{3}\right)\right)$ $\left\{ \frac{8\pi}{3} = 3\pi - \frac{\pi}{3} \right\}$

$= 2^8(2(-\cos \pi/3))$

$= 2^8\left(2\left(-\frac{1}{2}\right)\right) = -2^8 = -256$. Ans

b $(1+i\tan\theta)^n = \frac{\cos n\theta + i\sin n\theta}{\cos^n\theta}$

$LHS = (1+i\tan\theta)^4 + (1-i\tan\theta)^4$

$= \frac{\cos 4\theta + i\sin 4\theta}{\cos^4\theta} + \frac{\cos 4\theta - i\sin 4\theta}{\cos^4\theta}$

$= \frac{2\cos 4\theta}{\cos^4\theta}$

c let $z = i\tan\theta$.

so $(1+z)^4 + (1-z^4) = 0 \Rightarrow \frac{2\cos 4\theta}{\cos^4\theta} = 0$

$\Rightarrow 2\cos 4\theta = 0$

$\Rightarrow \cos 4\theta = \cos k\pi/2$

$4\theta = \frac{k\pi}{2}$

$\Rightarrow \theta = \frac{k\pi}{8}$

$\therefore z = i\tan\frac{k\pi}{8}$

d $(1+z)^4$

$^4C_0 1 + \cancel{^4C_1 z} + {^4C_2} z^2 + \cancel{^4C_3 z^3}$

$+ {^4C_4} z^4$

$+ {^4C_0} 1 - \cancel{^4C_1 z} + {^4C_2} z^2 - \cancel{^4C_3 z^3}$

$+ {^4C_4} z^4$

$1 + 6z^2 + z^4 + 6z^2 + z^4 + 1$

$= 2 + 12z^2 + 2z^4$

Now $2 + 12z^2 + 2z^4 = 0$

$\Rightarrow 1 + 6z^2 + z^4 = 0.$

$\Delta = b^2 - 4ac = 36 - 4\cdot(1)\cdot(1)$

$= 36 - 4 = 32.$

$z^2 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-6 \pm \sqrt{32}}{2}$

$z^2 = \frac{-6 + 2\sqrt{8}}{2}$

$z^2 = -3 + \sqrt{8}$

$z = \sqrt{-3 + 2\sqrt{2}}$